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FRACTIONAL MOMENTUM BALANCE EQUATION FOR A VARIABLE-MASS DYNAMICS

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Abstract

A new mechanical framework is introduced to describe bodies with time-continuous mass variation, incorporating a functional dependence on mass. In this approach, the dynamics are governed by momentum balance equations formulated using Caputo fractional derivatives, which adhere to a weak form of Galilean invariance. The formulation is particularly focused on the Meshchersky kinetics, accounting for both mass and velocity changes. As a practical example, this paper presents a novel model for the motion of a material body with continuously varying mass in a constant gravitational field—leading to a time-fractional version of the Tsiolkovsky rocket equation, augmented by a dissipative term. Under time-based approximation, deviations from vertical projectile motion are analyzed to assess the internal consistency of the proposed model.

Keywords: Variable-mass dynamics, Meshchersky equation, Tsiolkovsky rocket equation, Caputo fractional derivative

Introduction

During about fifty years or so, fractional calculus has attracted much attention due to its application in various fields of science and engineering. For various applications of fractional calculus in physics, mechanics etc., see [1-3] and references therein.

Two types of fractional derivatives or integrals, namely Riemann-Louville and Caputo derivative (both), are basics. Among other things, the Riemann-Louville fractional derivative of a constant is not zero, and it requires not generally specified fractional initial conditions. In contrast, Caputo derivative of a constant is zero, and a fractional differential equation expressed in terms of Caputo fractional derivative requires similar to standard boundary condition This is one of the reasons, why physicists or engineers prefer Caputo fractional derivative. In this 22 regard, are important papers [5-7]. In the work [S] investigated a failing body problem through the air in view of the fractional derivative approach, it has been demonstrated that using a dissipative term proportional to velocity is possible to use the Caputo derivative. Paper [6] contains general discuss about the fractional mechanics, where the time derivative is replaced with the fractional derivative, in the many cases: the motion of a body in a resisting medium where the retarding force is assumed to be proportional to the fractional velocity, also, fractional damped oscillator problem, the fractional harmonic oscillator problem, fractional forced oscillator problem etc. Authors in [7] carefully considered an improved version of the Boltzmann-Poisson model for **$BaTiO_3$-Ceramics**, lead to correct fractional relaxation meso mechanical velocity description in the case of time correlations. It is known that the classical mechanics is built upon the two concepts: inertial reference frames and Galilean invariance [8]. For materials described in the papers [5-7], the above mentioned does not have to be true [9]. In [9] starting from the Kac-Zwanzig Hamiltonian model generating Brownian motion, showed how Galilean invariance is broken during the coarse graining procedure when deriving stochastic equations, leads to a set of rules characterizing systems in different inertial frames that have to be satisfied by general stochastic models, which the authors have called weak Galilean invariance. In papers [S] and [7], phenomena, which are described, by construction, are within such a model.

Part of the mechanics, which describes the motion of the variable mass body, is a well-known [10], [11], [12]. In work [12] the Meshchersky equation is obtained on the basis of the equations of motion of mesoscopic particles. This premise led to the emergence of the components of acceleration connected with the internal processes in the body. However, neither the spatial nor temporal correlation of the corresponding physical processes was considered. In this paper, new mechanical variable-mass dynamics description of the body with time correlations continual functional dependence is considered. By assumption, in this case, dynamics of momentum balance described by Caputo fractional derivatives, which satisfy weak 33

Galilean invariance. Within this description, independently considering the kinetics of body velocity and mass.

Especially important is so called the Tsiolkovsky rocket model. For a Caputo fractional velocity failing body problem through the air, with dissipative term, in this paper, we study new Caputo fractional continual mass dynamics - the fractional Tsiolkovsky equation. The appropriate relation of balance of momentums is used. Thus the fractional Tsiolkovsky equation, in a special case, describes generalization of the model to the body moves in the neighborhood of the Earth's surface in the gravitational field, with constant velocity of separation of a certain mass and mass of the body decreases linear in the course of time [10], pp.110-113, in the case of weak dissipation. Within the time approximation of the body velocity, using the first deviation from the vertical projectile motion, the consistency of the given model is determined.

Mathematical preliminaries

We recall the following basic definitions and properties of fractional calculus theory which shall be used in this paper [1-4], [13].

Definition 1. A real valued function (t), t > 0 is said to be in the space C_{λ},$\boldsymbol{\lambda}$∈ R if there exists p > l, such that f(t) = $t^n f_1$ (t) where f_1 (t) ∈ C [0,∞) and it is said to be in the space C^n, ifand only if $f^n(t) \in C_{\lambda}$, n ∈ N.

Definition 2. The Riemann-Liouville left handed fractional integral operator of order $\alpha > 0$ of a function f(t) ∈ $C_{\lambda,}$ $\boldsymbol{\lambda}$>-1 are defined by:

$$ {}_{0}^{RL} I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0+}^{t} f(t') \, (t - t'))^{\alpha - 1} \, dt' $$

$$ {}_{0}^{RL} I^{0} f(t) = f(t') $$

By definition, $\Gamma(\alpha)$Euler's Gamma function. Let t ∈ [a, b] ⊂R and function f (t) is Lebesgue integrable in [a, b]: f (t) ∈ L([a,b]) Also, in this case it can be defined the left fractional Riemann-Liouville integrals by the previous equations in the form f${}_{0}^{RL} I^{\alpha} f(t)$, using replacement 0 → a in the lower bound of the integrals.

Especially important is the relation: if α→1, then $\Gamma(1-\alpha) \to \infty$. Basic properties of the Riemann-Liouville left handed fractional integral operator, for α,$\boldsymbol{\beta}$≥0, t > 0 and γ > -1 are:

$$ {}_{0}^{RL} I^{\alpha} \; {}_{0}^{RL} I^{\beta} f(t) = {}_{0}^{RL} I^{\alpha+\beta} f(t) $$

$$ {}_{0}^{RL} I^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma} $$

Definition 3. The left handed fractional derivative of (t) in Caputo sense is defined by relations:

$$ {}_{\alpha}^{C} I^{\delta} f(t) = \frac{1}{\Gamma(n-\delta)} \int_0^t dt' \, (t-t'))^{n-\delta-1} D^n f(t) \,|_{t=t'} $$

$$ D^n f(t) = \frac{d^n f(t)}{dt^n} $$

for n-l< δ ≤n, n ∊ N, t>0, f (t) ∊ C_{λ} , λ > -1. Knowing that the Caputo derivatives is described over the Riemann-Liouville integrals, and also for them is possible form ${}_{\alpha}^{C} D^{\delta} f(t)$. Basic properties of the Caputo operators, for $\boldsymbol{\alpha}$,**β**>0, t > 0, n ∊ N,are:

$$ {}_{0}^{C} D^{n} f(t) = \frac{d^n f(t)}{dt^n} $$

$$ c \in R \Rightarrow {}_{0}^{C} D^{\alpha} \quad c = 0 $$

$$ {}_{0}^{C} D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} $$

Theorem 1. let 0 < α ≤ l, and assume that f(t) and g(t) are analytic functions on (a-h, a + h). a≥0. Then,

$$ {}_{a}^{C} D^{\alpha}[f(t).\ g(t)] = \frac{t-a^{-\alpha}}{\Gamma(1-\alpha)}.g(a+).f(t) - f(a+) + f(t).{}_{a}^{C} D^{\alpha}[g(t)] + \sum_{k=1}^{\infty} \binom{\alpha}{k} D^k [f(t)].\ {}_{a}^{RL} D^{k-\alpha}[g(t)] \qquad (1) $$

At this point, two commentaries are necessary. First, we be kept in mind equation

$$\delta(\text{ t-a}) = \lim_{\alpha\to 1}\frac{\text{t}- a^{-\alpha}}{\Gamma(1-\alpha)} \qquad (2)$$

which is in accordance with Definition 2 In addition, for $\alpha > 0$ [1], valid

$$\begin{matrix} C \\ 0 \end{matrix} D^{\alpha}\sum_{k=0}^{n}\frac{t^{\alpha-j-1}}{\Gamma(\alpha-j)} = 0\,,\ C_j \in \text{R} \qquad (3)$$

In the further discussion, only those sets of functions that avoid the difficulties listed in the two previous relations are circumvented (it is assumed that itis possible), and, in the case when a = 0.

Standard variable-mass dynamics and Tsiolkjvsky model with dissipation

Within the mechanics, one known, the classical problem that is considered, according to [10-12] for m= m(t) and v = v(t) is mass and velocity of the body, $v_{rel} = v_{rel}(t) = v - v_e$, $v_e = v_e(t)$ is the velocity of the mass added to the exhaust, v_{rel} is the relative velocity of the escaping or incoming mass with respect to the center of mass of the body, F_{ext} is the net external force on the body, can be written in the form of the following equation of the momentum balance (Meshchersky equation):

$$\frac{d\,m.v}{dt} - \frac{\text{dm}}{dt}.v = \frac{\text{d}\ v_{rel}\,.m}{dt} - \frac{\text{d}\ v_{rel}}{dt}.m + F_{ext} \qquad (4)$$

The reason for this decomposition in the variable-mass dynamics for both sides of equation (4) is: from the equation of the time balance of the momentum, for concrete dynamics, subtracted an irrelevant term.

For constant velocity of separation of a certain mass v_e =constant, the Earth's surface in the gravitational field g, Tsiolkovsky equation with dissipative term b_1 v ($b_1 > 0$), is:

$$\frac{d\,m.v}{dt} - \frac{\text{dm}}{dt}.v = \frac{\text{d}\ v_e - v.m}{dt} - \frac{\text{d}\ v_e - v}{dt}.m - \text{mg} - b_1 v \qquad (5)$$

Equation (5) is written in the final form:

$$m.\frac{\mathrm{d}v}{dt} = (v_e - v).\frac{\mathrm{d}\,m}{dt} - \mathrm{mg} - b_1 v \qquad (6)$$

The conditions under which equation (6) is solved are:

$$V(0+) = v_0\,, m(t) = m_0 - \mu_1 . t\,, m(0+) = m_0$$

$$b_1 \geq 0,\ \mu_1 > b_1 \qquad (7)$$

The solution of the equation (6) is:

$$v = \frac{m_0 \cdot (v_0 - \frac{m_0\, g}{2\mu_1 - b_1} - \frac{\mu_1\, v_e}{\mu_1 - b_1})}{(1 - \frac{\mu_1}{m_0}.t)^{\frac{\mu_1 - b_1}{\mu_1}}} + \frac{g\, m_0 - \mu_1 . t}{2\mu_1 - b_1} + \frac{\mu_1 \cdot v_e}{\mu_1 - b_1} \qquad (8)$$

If for the first member on the right side of the equation (8) compute the Maclaurin polynomial of degree 2, we have:

$$v = m_0 + m_0^{\frac{b_1}{\mu_1}}.\left(\frac{\mu_1 - b_1}{m_0} \quad v_0 - \frac{m_0\ g}{2\,\mu_1 - b_1} - \frac{\mu_1\ v_e}{\mu_1 - b_1}\right).\left(1 + \frac{\mu_1 - b_1}{m_0}\ t + \frac{(\mu_1 - b_1)\,(2\mu_1 - b_1)\,(v_0 - m_0\ g)}{2\,m_0^2}\ \mathrm{t}^2\right) + \frac{m_0\ g}{2\,\mu_1 - b_1} - \mu_1 t + \frac{\mu_1\ v_e}{\mu_1 - b_1} \qquad (9)$$

For $\mu_1 \to 0$, equation (9) expected tends to v = v_0 – gt. The consistency of this model can also be demonstrated in the following way. Considered the second order polynomial expansion

$$V(\mathrm{t}) = v_0 + A_1 . t + B_1 . \mathrm{t}^2 \qquad (10)$$

in eq. (6), if $\mu_1 \to 0$,, results are:

$$A_1 = \frac{(\mu_1 - b_1)\ v_0 - m_0\, g - \mu_1\ v_e}{m_0}$$

$$B_1 = \frac{(2\mu_1 - b_1)\,A_1 + \mu_1\,\mathrm{g}}{m_0} \qquad (11)$$

All asimptotics (especially for a member degree two t^2 - correction to free fall, in (6) for this polynomial expansion can be written in the form $-\mu_1\ B_1 - 0.5(\ \mu_1 - b)\,B_1$ are corrects. When $\mu_1 \to 0$, the equation (9) expected converges to equation (10).

Basic idea-assumption for the fractional variable- mass dynamics presents in this paper is: mesoscopic or macroscopic dynamics of momentum balance described using Caputo fractional derivatives, which satisfy weak Galilean invariance, bearing in mind the equations (1) and (4). Result, for 0<α,**β**≤1 is time fractional Meshchersky equation:

$$t_v{}^{\alpha-1}.\left(\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\ .v_0.mt\right)-\ m_0+\left(mt.{}_0^C\ D^{\alpha}[vt]\ \right)=\right.$$
$$t_m{}^{\beta-1}.\left(\left(\frac{t^{-\beta}}{\Gamma(1-\beta)}\ m_0\ .v_0\right)-vt-\ v_e-\left(vt.{}_0^C\ D^{\beta}[mt]\ \right)+\ F_{ext}\right.$$
(12)

where, constants t_v and t_m are a fractional time relaxations, respectively, for velocity and mass of the body (see First relaxation time t_v , by [5], represents atmospheric characteristics, while tm is a fuel property. The stated characteristic and properties are to be independent. In work [5] , in equation (10), there is no term of type $\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, which is a result of Theorem 1. It exists in this paper, as a consequence of broken Galilean invariance, although it may be omitted from the technique shown in [10], or reformulation of the Caputo fractional derivative. The method presented by the equations (10) and (11), which is used below, would remain the same. Rearranging fractional Newton's second law for the motion equation variable-mass body dynamics, when considering the fractional balance of the velocity and mass is in the spirit of Eq. (4) (i.e. removal unnecessary terms).

Then, Caputo fractional generalization Eq. (5), if α = **β**, b_α = constant, in (10), is:

$$t_v{}^{\alpha-1}.\left(\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\ .v_0.mt\right)-\ m_0+\left(mt.{}_0^C\ D^{\alpha}[vt]\ \right)=\ t_m{}^{\alpha-1}.\left(\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\ m_0\ .v_0\right)-\right.\right.$$
$$vt-\ v_e-\left(vt.{}_0^C\ D^{\alpha}[mt]\ \right)-mtg-\ b_\alpha v \qquad (13)$$

Using the given equation, a fractional Tsiolkovsky model is described. Some solutions of this equation will be specially considered. Previous equation, for μ_α =constant, satisfies the new conditions:

$$mt = m_0 - \mu_0 \cdot \frac{t^\alpha}{\Gamma(1+\alpha)} \qquad (14)$$

$$\mu_0 > b_\alpha \; t_m{}^{1-\alpha}$$

Then (14), multiply by t^α, is

$$k_{vm\alpha} \cdot \left(-\frac{\mu_\alpha v_0 t^\alpha}{\Gamma(1-\alpha)\,\Gamma(1+\alpha)} + \left(m_0 t^\alpha - \frac{\mu_\alpha t^{2\alpha}}{\Gamma(1+\alpha)}\right) {}_0^C D^\alpha v \right) = \left(-\frac{m_0}{\Gamma(1-\alpha)} + \mu_\alpha - b_\alpha t^\alpha \; t_m{}^{1-\alpha}\right) v - v_e \, \mu_\alpha t^\alpha - \left(m_0 t^\alpha - \frac{\mu_\alpha t^{2\alpha}}{\Gamma(1+\alpha)}\right) t_m{}^{1-\alpha} g + \frac{m_0 v_0}{\Gamma(1-\alpha)} \qquad (15)$$

where is:

$$k_{vm\alpha} = \left(\frac{t_v}{t_m}\right)^{1-\alpha} \qquad (16)$$

The solution of the equation (15), considered asymtotics $\mu_\alpha \to 0$, is calculated in the form:

$$v = v_0 + A_\alpha \frac{t^\alpha}{\Gamma(1+\alpha)} + B_\alpha \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \qquad (17)$$

Then, for A_α and B_α results are:

$$A_\alpha = \frac{\left(\mu_\alpha \left(1 + \frac{k_{vm\alpha}}{\Gamma(1-\alpha)\,\Gamma(1+\alpha)}\right) - b_\alpha \; t_m{}^{1-\alpha}\right) v_0 - \mu_\alpha \; v_e - m_0 \; t_m{}^{1-\alpha} g}{m_0 \left(k_{vm\alpha} + \frac{1}{\Gamma(1-\alpha)}\right)} \qquad (18)$$

and

$$B_\alpha = \frac{2\mu_\alpha - b_\alpha \; t_m{}^{1-\alpha} \; k_{vm\alpha} A_\alpha + \mu_\alpha \; t_m{}^{1-\alpha} g}{m_0 \, \Gamma(1) + \alpha \, k_{vm\alpha} \left(\frac{1}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Gamma(1+2\alpha)}\right)} \tag{19}$$

The coefficient of $t^{3\alpha}$ then can be written in the form If , (18) and Eq. (19) tends to (11). It should be noticed, also: all asimptotics are correct Constant Ba showing a variable mass correction constant, with the case of weak dissipation, for deviation from vertical projectile motion in the constant Earth 's gravitational field.

Basic concepts of mesoscopic physics systems in the paper [14] are: quantum coherence (one-particle wave-functions approximation), quantum transport, disordered and ballistic systems, samples as 'doubly open" quantum systems, quantum chaos, or correlations, fractals and levy flight (or similar motions) [7]. In the opposite, author in the work [12], describe the equation of Meshersky (S) for motion of a mesoscopic particles considering their sizes, the presence of internal structure and internal processes in them, without taking into account explicitly these quantum and others effecG. Of course, it should be noted that exists also macroscopic many quantum phenomena [IS]. Regardless ofthe size of the particles, this paper introduces time correlations of the atmosphere and fuel on its velocity of their motions, and, indirectly, between them. The introduction of correlations contributes to the foundation of the suppositions and practices of a body variable mass dynamics in the spirit of the solid state theory [1 S]. An example of a macroscopic quantum phenomenon at room temperature given in [16]. In addition to the paper [5], this circumstance suggests that spatial-temporal correlations are possible at higher temperatures.

Conclusion

The macroscopic mechanical problem of a moving body with variable mass is both classical and fundamentally important. A particularly notable case is the Tsiolkovsky rocket equation, which plays a key role in optimizing parameters for the operation and control of rocket propulsion systems. Building upon this foundation, further generalizations of equation (12) are conceivable.

One such direction involves incorporating space-time correlations in the mechanics of mesoscopic particles, akin to those explored in [12]. Another involves accounting for strong correlations between external processes and mass variation dynamics, potentially within frameworks such as plasma analogies [15].

The fractional model of variable-mass dynamics presented in this paper offers a novel approach to capturing these effects. If validated experimentally—either directly or in conjunction with models like those proposed in [12] or [15]—this framework could represent a meaningful advancement in both scientific understanding and technological development.

References

[1] Voit E O. Biochemical Systems Theory: A Review. Isrn Biomath 2013;897658:53.

[2] Magin R L. Fractional Calculus in Bioengineering. Begell House, Redding, Calif, USA, 2006.

[3] Xia Y, Han M. Multiple periodic solutions of a ratio-dependent predatorprey model. Chaos Solitons Fract 2009;39:1100-1108.

[4] Yang J, Zhao L. Bifurcation analysis and chaos control of the modified Chua's circuit system. Chaos Solitons Fract 2015;77:332-339.

[5] Bohner M, Peterson A. Dynamic equations on time scales. Birkhäuser, Boston, 2001.

[6] Bohner M, Peterson A, Eds. Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, 2003.

[7] Atici F M, Ş engül S. Modeling with fractional difference equations. J Math Anal Appl 2010;369(1):1-9.

[8] Savageau M A. Michaelis-Menten mechanism reconsidered: implications of fractal kinetics. J Theor Biol 1995;176(1):115-124.

[9] Schnell S, Mendoza C. Closed form solution for time-dependent enzyme kinetics. J Theor Biol 1997;187(2):207-212.

[10] Meena A, Eswari A, Rajendran L. Mathematical modelling of enzyme kinetics reaction mechanisms and analytical solutions of non-linear reaction equations. J Math Chem 2010;48(2):179- 186.

[11] Goh S M, Noorani S M, Hashim I. Introducing variational iteration method to a biochemical reaction model. Nonlinear Anal-Real 2010;11(4):2264-2272.

[12] Hashim I, Chowdhury M S H, Mawa S. On multistage homotopy-perturbation method applied to nonlinear biochemi- cal reaction model. Chaos Solitons Fract 2008;36(4):823-827.

[13] Alawneh A. Application of the Multistep Generalized Differential Transform Method to Solve a Time-Fractional Enzyme

Kinetics. Discrete Dyn Nat Soc 2013;592938:7.

[14] Kilbas A A, Srivastava H M, Trujillo J J. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam, The Netherlands, 2006.

[15] Chen CK, Ho SH. Application of differential transformation to eigenvalue problems. Appl Math Comput 1996;79:173-188.

[16] Chen CK, Ho SH. Solving partial differential equation by differential transformation. Appl Math Comput 1999;106:171-179.

[17] Odibat Z, Momani S, Ertürk V S. Generalized differential transform method: application to differential equations of fractional order. Appl Math Comput 2008;197(2):467-477.

[18] Ertürk V S, Odibat Z M, Momani S. An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^+$ T-cells. Comput Math Appl 2011;62:996-1002.

[19] Odibat Z M, Shawagfeh N T. Generalized Taylors formula. Appl Math Comput 2007;186:286-293.

[20] Odibat M Z, Bertelle C, Aziz-Alaoui M A, Duchamp G H E. A multi-step differential transform method and application to non-chaotic or chaotic systems. Comput Math Appl 2010;59(4):1462-1472.

[21] Momani S, Odibat Z. A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula. J Comput Appl Math 2008;220:85-95.

[22] Abdeljawad T. On Riemann and Caputo fractional differences. Comput Math Appl 2011;62:1602-1611.

[23] Čermák J, Nechvátal L. On (q, h)-analogue of fractional calculus. J Nonlinear Math Phys 2010;17(1):51–68.

[24] Rahmat M R S, Noorani M S M. Caputo type fractional difference operator and its application on discrete time scales. Advances in Difference Equations 2015;2015:160.

[25] Ferreira R A C, Torres D F M. Fractional h-difference equations arising from the calculus of variations. Appl Anal Discrete Math 2011;5:110-121.

[26] Atici F M, and Eloe P. W. A Transform method in discrete fractional calculus. IJDE 2007;2(2):165-176.

[27] Samko S G, Kilbas A A, Marichev O I. Fractional Integrals and Derivatives, Theory and Applications. Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

[28] Čermák J, Kisela T, Nechvátal L. Discrete Mittag-Leffler func- tions in linear fractional difference equations. Abstr Appl Anal 2011;565067:22.

[29] Jaroszkiewicz G. Principles of Discrete Time Mechanics. University Printing House, Cambridge CB2 8BS, United Kingdom, 2014.