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THE FRACTIONAL TSALLIS ENTROPY

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Abstract

A three-parameter generalization of the Tsallis entropy based on the properties of the power functions and Weyl fractional calculus like extension of quantum calculus, are introduced. The generalization of the Shannon-Khinchin axioms corresponding to the fractional Tsallis entropy is verified and proposed. These axioms uniquely characterize new entropy function. For a certain sets of parameter values satisfied the second and third law of thermodynamics, the Lesche and thermodynamic stability criteria.

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1. Introduction

Statistical entropy is a measure of the number of possibilities available to a system, and assume its minimally zero when the system is in a given state and maximal value when a system can be in a number of micro states randomly with equal probability, with no uncertainty in its description. Over the past three decades, there has been a lot of interest in generalizing the Shannon entropy and exploring the consequences of applying these new concepts in several scientific fields[1-12]. For the new entropy functions are considered properties characteristic for the Shanon entropy: non-negativity, additivity, monotonicity and continuity, extensivity, convexity, stability, and, particularly, whether they conform to the second and third law of thermodynamics[13-14]. As a consequence of the mentioned, central tendency to the development of the statistical mechanics of systems is the definition of the free energy. The existence of this function is the result of normalization of the probability distribution function, which in turn controls the behavior of all the macroscopic properties of the ensemble. The majority entropy functions depend on an additional parameter q and become the Shannon entropy function when this parameter takes the value $q = 1$. These generalizations mostly could be non-extensive and opening the possibility for applications to systems with long range interactions between meso or macroscopic parts of system and non-additivity of energies also on meso and macroscopic scales.

Fractional calculus (FC) is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional

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exponents is an outgrowth of exponents with integer value. FC is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. Precise mathematical formulation of basic fractional calculus or its many applications are given in Refs. 15-17. Quantum q-calculus presented in Ref. 18, having a close connection with commutativity relations in the Lie algebras and is particularly useful for the quantum groups. Possible applications q-calculus are in geometry over finite fields.

In papers known expressions for entropy inspired in the properties of basic FC[11-13] .Order of derivative operators in FC is a strong connected to entropic parameter. In the statistical mechanics, the main motivation to propose new entropies to be able to describe phenomena that lie outside the scope of the Boltzmann–Gibbs (BG) formalism.

In the present paper, introduced a new entropy function based on the properties on the power functions, FC and q-calculus. For that purpose is notice that the Shafee entropy[7] and the Ubriaco entropy[11] can natural generalize into two-parameter concept. After that, considers properties of expansion mentioned two-parameter entropy in the sense of the Tsallis entropy.

Letter is organized as follows. After the introduction in section 1, in next section introduced some necessary definitions and mathematical preliminaries of FC. In section 3, derived the new entropy functions. There is demonstrated that they are related to the new nonlinear operator of fractional derivative. In section 4, described some properties of this entropies. Finally, section 5 outlines the main conclusions.

2. Preliminaries and notations

In the literature exists various definitions of fractional order derivatives. One of these definitions of a fractional order derivative is the Weyl definition.

The Weyl fractional derivative order q is defined as[15,17]

$$\left({}_W D_t^q f\right)(t) := \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{-\infty}^{t} dt' \cdot \frac{f(t')}{(t-t')^{q+1-n}},\ n-1 < q \le n,\ n \in \mathbf{N}. \tag{1}$$

For the Weyl fractional derivative valid relation[11]

$${}_W D_t^q e^{\lambda \cdot t} = \lambda^q e^{\lambda \cdot t}, \tag{2}$$

where $q > 0$ and $\lambda > 0$. This derivative is a linear operator.

3. Establishing to the new entropy concept

The Tsallis entropy observation can be defined from the equation[1,9,19]

$$S_{Tq} = \lim_{t \to -1} D_q^t \sum_i p_i^{-t}, \tag{3}$$

opened the possibility to define new entropy functions[20] . Where the operator $D^t{}_q$ is called the Jackson q-derivative[21] defined as

$$D_q^t f(t) := t^{-1} \frac{f(tq) - f(t)}{q-1}. \tag{4}$$

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Parameter q have a positive real values and sometimes called entropic index, p_i is the probability distribution function (pdf). Eq. (4) usually written in the form[22]

$$S_{Tq} = \frac{1}{1-q}\left(\sum_i p_i^q - 1\right). \tag{5}$$

In the limit $q \to 1$ the Shanon entropy recovered. Earlier to Tsallis entropy, was introduced an additive entropy - Rényi entropy. Between first or true entropies, the Hartley and Shannon entropies, both Rényi and Tsallis entropies are interpolation formulas. They are connected by the equation[19] $(q-1)S_{Tq} + \exp\left[-(q-1)S_{Rq}\right] = 1$, where S_{Rq} is the Rényi entropy. Sometimes, these parametric form of entropies are criticized on the ground that they are not true entropies, and therefore the same is true for their generalizations. The basic reason is that they depend on a parameter, which is different for different systems. After these, many physically different definitions of entropy can be given, and what makes up a "physically relevant entropy" is often subject to a lot heated discussions. As opposed to previous one, exists is a conception that the BG statistics cannot yield the long tail distribution. This is the justification for the intensive research of nonextensive and other entropies[23,24,25].

Ubriaco[11] proposed entropy functions based on FC which has a physical sense[13]

$$S := \lim_{t \to -1} {}_W D_t^q \sum_i e^{-t \cdot \ln p_i},\ q \in (0, \infty). \tag{6}$$

In another form Eq. (6) can be written as follows

$$S_q = \sum_i p_i \cdot \left(-\ln(p_i)\right)^q \tag{7}$$

The new and direct generalization of the Eq. (6) is

$$S_{q_1,q_2} := \lim_{t \to -q_1} {}_W D_t^{q_2} \sum_i e^{-t \cdot \ln p_i}. \tag{8}$$

That the entropy becomes the function

$$S_{q_1,q_2} = \sum_i p_i^{q_1} \cdot \left(-\ln(p_i)\right)^{q_2},\ q_1, q_2 > 0. \tag{9}$$

If $q_1 = q$ and $q_2 = 1$, then Eq. (9) described the Shafee entropy function[7]. Entropy (9) is examined in detail in Ref. 26. This entropy is a concave function for positive q_1 and q_2. The Shafee entropy functions in Ref. 7 defined by mixing probability $M(q) = 1 - \sum_i p_i^q$: $S(q) = -\frac{dM}{dq}$. $M(q)$ represents, in the cell-letter model, the measure of the disorder, introduced by increasing the cell scale from $q = 1 + \Delta q$ to $q = 1$ or $q > 1$ ($\Delta q > -1$). The introduction of fractional values of cell numbers can be taken in the same spirit as defining the fractal (Hausdorff) dimensions of dynamical attractors and in complex systems[27,28]. Therefore, with the above, entropy (9) can be considered in the Ref. 26 as a fractional entropy in a fractal phase space in which the parameter q_1 comes from the fractal nature and the parameter q_2 is from the fractional aspect. However, as was noted on the basis of (8), Eq. (9) is completely

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derived from the FC. This means that the t-axis describes fractional values of cell numbers. If it is accepted limit such in Eq. (8) , but in Eqs. (3) and (4), can be written

$$S_{q;q_1}=\frac{\sum_i p_i^{q_1}\left(p_i^{(q-1)q_1}-1\right)}{q_1(1-q)}. \tag{10}$$

Then

$$\lim_{q\to 1} S_{q;q_1}=-\sum_i p_i^{q_1}\cdot\ln(p_i). \tag{11}$$

Previous equation gives an idea that is based Tsallis entropy writings such a function that when $q \to 1$ will be (9). The simplest option for generalizations in this sense, due to using the power functions under the sum in Eq. (10), is appropriate degree expression of its right side. This idea is based to this circumstance that the concept of usage some power functions for the entropy first time directly used in the Ref. 7. Should also noted that the Tsallis and Rényi entropies based on the profities of the power functions. Then explicitly obtained three parametric entropy function

$$S_{q;q_1,q_2}[p]=\left\{\frac{\sum_i\left(p_i^{\frac{qq_1}{q_2}}-p_i^{\frac{q_1}{q_2}}\right)}{\frac{q_1(1-q)}{q_2}}\right\}^{q_2}. \tag{12}$$

In the previous equation, $S_{q;q_1,q_2}[p]=\sum_i \phi_{q;q_1,q_2}(p_i)=\sum_i s_{iq;q_1,q_2}(p)$ is a continuous positive functions, which is valid

$$S_{q;q_1,q_2}[p]=\left(\frac{q_2}{q_1}\right)^{q_2}\left(\left(\frac{1-qq_1q_2^{-1}}{1-q}\right)S_{T\frac{qq_1}{q_2}}[p]+\left(\frac{q_1q_2^{-1}-1}{1-q}\right)S_{T\frac{q_1}{q_2}}[p]\right)^{q_2}, \tag{13}$$

For $q_1=q_2=1$, (13) describe the Tsallis entropy S_{Tq} and for $q_1=q_2\neq 1$, two-parameter entropy is the Ubriaco like case. Taking into account the Eq. (3), introduced a new fractional q_2 - Weyl like, q - derivative

$${}_{W}^{q_2}D_q^t f(t):=\left\{\frac{\left(f\left(\frac{q}{q_2}t\right)-f\left(\frac{t}{q_2}\right)\right)}{\frac{t(q-1)}{q_2}}\right\}^{q_2},\ q_2\neq 0. \tag{14}$$

If $q_2=0$, then, by definition, ${}_{W}^{0}D_q^t f(t):=\infty$. For $q_2=1$ then (14) becomes (4). If perform substitution $\frac{t}{q_2}\to t$ in Eq. (14), obtained the relation $\left(D_q^t f(t)\right)^{q_2}$. Easy observed that the fractional q_2 - Weyl like, q - derivative is not a linear operator. Both derivatives, fractional and q-derivative, have their physical meaning and the combination thereof (14) may useful.

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However, the main conclusion is that for the entropy (12) dominant influence does the Tsallis entropy.

4. Entropy and its axiomatic characterization

For the Tsallis entropy exists various experimental verifications and applications[1,14]. The Ubriaco[9,29], Shafee[27,28] and entropies gives by Eq. (9) haven't so much experimentally verifications. Still, assumed that entropy (12) discussed here may be a strong candidate in describing complex system.

Below, presented the three-parameter generalization of the Shannon-Khinchin axioms in Abe sense[1,30]. Let Δ_n be an n-dimensional simplex

$$\Delta_n := \left\{ \mathbf{p} = \left(p_1, ..., p_n\right) \mid p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}. \tag{15}$$

For Tsallis entropy, let $\mathbf{p}^A \in \Delta_n^A$ and $\mathbf{p}^B \in \Delta_n^B$ be two probability distributions for two subsystems A and B respectively. Then the entropy of composite system is

$$S_{Tq}\left[A,B\right] = S_{Tq}\left[A\right] + S_{Tq}\left[B\right] + \left(1-q\right) S_{Tq}\left[A\right] S_{Tq}\left[B \mid A\right]. \tag{16}$$

If A and B is independent, hence $S_{Tq}\left[B \mid A\right] = S_{Tq}\left[B\right]$. The Tsallis entropy is a pseudo-additive. Three-parameter entropy (12) satisfies the following axioms.

(i) Continuity. The entropy $S_{q;q_1,q_2}$ is continuous in Δ_n . Previous restrictions are the result of the basic principles of thermodynamics which will later prove.

(ii) Maximality. For any $n \in \mathrm{N}$, then valid

$$S_{q;q_1,q_2}\left(\mathbf{p}\right) \leq S_{q;q_1,q_2}\left(\frac{1}{n}, ..., \frac{1}{n}\right). \tag{17}$$

(iii) Expansibility. Consider

$$S_{q;q_1,q_2}\left(\mathbf{p}, 0\right) = S_{q;q_1,q_2}\left(\mathbf{p}\right). \tag{18}$$

(iv) Generalized Shannon Additivity. Consider

$$S_{q;q_1,q_2}\left[A,B\right] = \left(\frac{q_2}{q_1}\right)^{q_2} \left(\left(\frac{1 - q q_1 q_2^{-1}}{1-q}\right) S_{T\frac{q q_1}{q_2}}\left[A,B\right] + \left(\frac{q_1 q_2^{-1} - 1}{1-q}\right) S_{T\frac{q_1}{q_2}}\left[A,B\right] \right)^{q_2}, \tag{19}$$

including Eq. (16).

If the above four requirements are hold, entropy (12) defined in a unique way. All of these properties easy derived from characteristics of Tsallis entropy[1,30] and Eq. (13).

It is essential, however, to put the fundamental restrictions on values of the parameters q, q_1 and q_2. First, obvious that $S_{q;q_1,q_2}[p]$ is a concave function because $\left(S_{q;q_1,q_2}[p]\right)^{q_2^{-1}}$ is a concave function due (13) (the Tsallis entropy is a concave function for positive q). It is clear that they are exact relations $q q_1 q_2^{-1} > 0$ and $q_1 q_2^{-1} > 0$. Maximum of function (12) established for

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$p_i = q^{\frac{q_2}{q_1(1-q)}}$. If $q \to 1$ then $p_i \to \exp\left(-\frac{q_2}{q_1}\right)$. Second, by definition, $s = s_{iq;q_1,q_2}(p)$ must be bounded functions and, consequently, for $q_2 > 0$, they have constrained values.

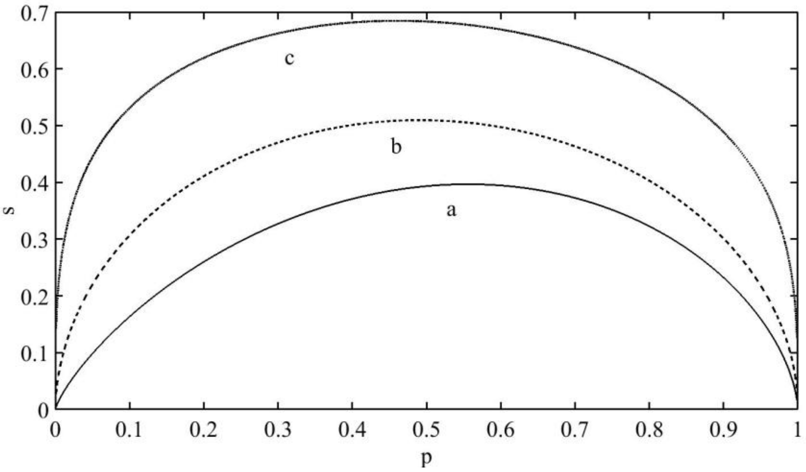

Fig. 1. Plots of the $s = s_{iq;q_1,q_2}$ function versus p for: a) q=1.6, q_1=0.8, q_2=0.6; b) q=1.6,q_1=0.55,q_2=0.5; c) q=1.6, q_1=0.303, q_2=0.3.

The functions $s = s_{iq;q_1,q_2}(p)$ which illustrate in **Fig. 1**, are typical positive concave functions.

If considered the continuum case $n \to W \geq 1$, then Eq. (17) produced the relation

$$S_{q;q_1,q_2}\left[\frac{1}{W}\right] = W\left\{\frac{\left(W^{-\frac{qq_1}{q_2}} - W^{-\frac{q_1}{q_2}}\right)}{\frac{q_1(1-q)}{q_2}}\right\}^{q_2} . \quad (20)$$

For the Eq. (20) can be concluded that for $0 < q \leq 1$ and $0 < qq_1/q_2 \leq 1$or $q \geq 1$ and $q_1/q_2 \leq 1$, $S_{q;q_1,q_2}$ is a nondecreasing concave function of W which regular diverges ("as a power law") if $W \to \infty$ (see **Fig. 2**) as similar function for Tsallis entropy if $q < 1$. In opposite case, for $0 < q \leq 1$ and $0 < q_1/q_2 \leq 1$or $q \geq 1$ and $qq_1/q_2 \leq 1$, $S_{q;q_1,q_2}$ have a horizontal asimptote, as similar function for Tsallis entropy if $q > 1$ (see **Fig. 2**). The concavity of the entropic function illustrated through a set of plots shown above in **Fig. 1**. The aforementioned conclusions will be used in the next section.

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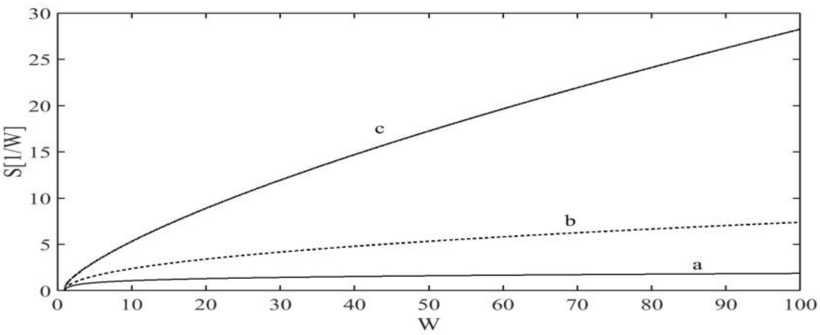

Fig. 2. Plots of the function $S_{q;q_1,q_2}[1/W]$versus W for: a) q=1.6, q_1=0.8, q_2=0.6; b) q=1.6, q_1=0.55, q_2=0.5; c) q=1.6, q_1=0.303, q_2=0.3.

In **Fig. 2**, presented plots o*f* $S[1/W]$ for some values of q, q_1, q_2 (the generalized pdf). Clearly in the limit $q, q_1, q_2 \to 1$ the Boltzmann relation $S = ln\ W$ is recovered.

5. Thermodynamic properties

The study of the stability properties of entropy functions is one of the important issues that need to pay attention to many works. In the framework of the above, Lesche, in a pioneering articles[31,32], proposed a criterion to study the stability of the Rénya entropy function and BG entropy. For this criterion the motivation can be formulated as follows. The basic motive for existence of this type of stability is to check whether existence of quantitative sensitivity to changes when the probability assignments p on a set of n microstates is perturbed by an infinitesimal amount δp (i.e. experimental robustness). To some generalizations of the Shannon entropy, this criteria has already been applied[31,32-36]. For Tsallis entropy is shown that Leshe stable[37]and the same holds for $S_{q;q_1,q_2}[p]$due to Eq. (13). Such is the entropy of the given in Eq. (9)[26]. In the case of thermodynamic stability, features consideration is different. Condition in thermodynamic stability of system in the BG formalism, as is well known, is equivalent to the concavity of the entropy: $\frac{\partial^2 S}{\partial E^2} < 0$, E is the internal energy of the ensemble per constituent. It should be noted that in the case of a non-additive entropies, the property of concavity does not imply thermodynamic stability[35].

The Tsallis entropy is the thermodynamic stable for $0 < q < 1$[38], until entropy functions given by Eq. (9) are stable in this way for $0 < q_1 < 1$ and $q_2 > log_2\, q_1$ or $q_1 > 0$ and $q_2 < log_2\, q_1$[26].

Generally considered, the Tsallis statistics, by investigating the second law of thermodynamics in the context of kinetic theory, has been studied in the classical[39], the relativistic[40] , and also in the quantum-mechanical regimes[41]. Another study discussed for the generalized relative entropy the convexity property in the quantum regime[42], leading for the Tsallis entropy to the constraint $0 < q \leq 2$. Putting together results for third law of

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thermodynamics reported in[14] $q \geq 1$, may conclude that the Tsallis entropy S_{Tq} is compatible with all the laws of thermodynamics only for q in the range $1 \leq q \leq 2$.

Entropy $S_{q;q_1,q_2}[p]$therefore compatible with all the laws of thermodynamics, due to (13), if $1 \leq qq_1q_2^{-1} \leq 2$ or $1 \leq q_1q_2^{-1} \leq 2$.

Then, for $0 < q \leq 1$ and $0 < qq_1/q_2 \leq 1$or $q \geq 1$ and $q_1/q_2 \leq 1$, concluding relations are

$$0.5 \leq q < 1,\ \frac{q_1}{q_2} = \frac{1}{q}, \qquad (21a)$$

$$1 < q \leq 2,\ \frac{q_1}{q_2} = 1 \qquad (21b)$$

and $q_2 \leq 2$. Relations (21a) and (21b) represents some of the main results of this paper. In the cases if for $0 < q \leq 1$ and $0 < q_1/q_2 \leq 1$or $q \geq 1$ and $qq_1/q_2 \leq 1$, results are $q_1q_2^{-1} = 1$ and $q = 1$. Previously mentioned inequalities represent physical limitations to the entropy function given in Eq. (9) and Ref.26. If $qq_1q_2^{-1} = 1$ or $q_1q_2^{-1} = 1$then $S_{q;q_1,q_2}[p]$ presents a two-parameter Tsallis like entropies.

Based on the statistical-thermodynamic principles, the probability distributions can be obtained by maximizing the corresponding entropy function $S_{q;q_1,q_2}[p]$(see Refs. 7, 26 for more details), under the constraints $\Sigma_i\, p_i = 1$ and $\Sigma_i\, p_i\, \varepsilon_i = E$ (ε_i is the i-th state energy), subject to constraint equation

$$L_{q;q_1,q_2} = S_{q;q_1,q_2}[p] + \alpha\left(1 - \sum_i p_i\right) + \beta\left(E - \sum_i \varepsilon_i p_i\right) \qquad (22)$$

where α and β are the Lagrange multipliers associated with the normalization of the pdf's p_i and the conservation of energy, such that setting $\frac{\partial L_{q;q_1,q_2}}{\partial p_j} = 0$, leads to the equation

$$\phi_{q;q_1,q_2}{}'(p_i) = \alpha + \beta\varepsilon_i \qquad (23)$$

with the solution $\phi_{q;q_1,q_2}(p_i) = (\alpha + \beta\varepsilon_i)p_i + C$, where C is the constant of integration. Hence gets relations for the pdf's

$$p_i = \psi_{q;q_1,q_2}{}^{-1}\left(\beta(\varepsilon_i - A)\right) \qquad (24)$$

in accordance with the definition

$$\psi_{q;q_1,q_2}(p) := \frac{\phi_{q;q_1,q_2}(p)}{p} \qquad (25)$$

Function $A = -\alpha/\beta$ called the Helmholtz free energy. For Eq. (24) assumed that the function can be inverted. Generally, do not have a closed form for p_i solution for arbitrary q, q_1 and q_2.

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But one should be able to obtain the *pi* for given q, q_1 and q_2 either numerically or analytically.

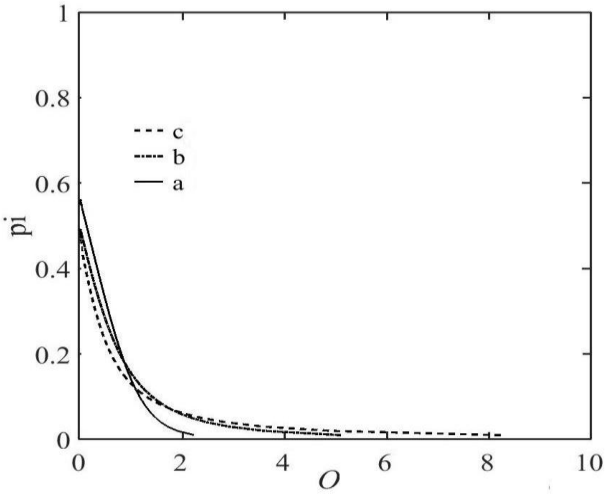

Fig. 3. Plots of the unnormalized pdf p_i of the quantity $O=\alpha+\beta\varepsilon_i \geq 0$ for: a) q=1.6, q_1=0.8, q_2=0.6; b) q=1.6, q_1=0.55, q_2=0.5; c) q=1.6, q_1=0.303, q_2=0.3.

In **Fig. 3** presents, for some values of q, q_1 and q_2 numerically determined plots of relation (23) using the variable $O=\alpha+\beta\varepsilon_i$. In all figures the specific values of q, q_1 and q_2 should come from real physical systems.

5. Conclusions

This paper presented a generalizations of the concept of entropy inspired in the properties on the power functions, FC and q-calculus. Within context of new calculus, defined a new entropy functions. This new entropies is concave, positive definite, non-additive, for given set of values of three parameters satisfies generalization of the Shannon-Khinchin axioms, stability criteria and the second and third law of thermodynamics. In the description of its properties dominate characteristics on the Tsallis entropy.

The relationship between fractional derivatives and entropy functions are only recently being considered. In the framework of the above are particularly interesting other fractional or different modification of Jackson q-derivative except one described in Ref. 19 or this article. Accordingly, it is very important to mention that Makhaldiani[43] presents his version of the fractional q-derivative

$$\left(D_q^t\right)^\alpha f(t) := \left((1-q)t\right)^{-\alpha}\left[f(t)+\sum_{n\geq 1}(-1)^n \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} f\left(q^n x\right)\right] \qquad (26)$$

in the context of algebra-analytic quantization and field theory. The fractional derivative defined by Eq. (26) is a very different from those introduced by (14). The Makhaldiani fractional q-derivative, as opposed to the operator (14), is, by definition, linear operator. His

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action on the functions, although he probably represents the better candidate for new entropy, however, is more complicated.

Towards this end, establishing a possible connection between these two operators, including applicability to the description of the concept of entropy of the last defined operator (26) or similar, could be the subject of future research.

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